Comment on the sign of the Casimir force

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Abstract

I show that reflection positivity implies that the force between any mirror pair of charge-conjugate probes of the quantum vacuum is attractive. This generalizes a recent theorem of Kenneth and Klich [9] to interacting quantum fields, to arbitrary semiclassical bodies, and to quantized probes with non-overlapping wavefunctions. I also prove that the torques on charge-conjugate probes tend always to rotate them into a mirror-symmetric position.

The Casimir force [1] and related electromagnetic interactions [2, 3] are the dominant forces between neutral non-magnetic objects at scales ranging from a few nanometers to hundreds of microns. These forces play therefore an important role both in mesoscopic gravity experiments [4], and in the design and fabrication of micro- and nano-electromechanical systems [5, 6]. Of particular interest is the possibility of repulsive interactions, since a frequent cause of malfunction of miniature devices is the collapse and subsequent permanent adhesion of mechanical elements, known as 'stiction' [7, 8]. It is therefore important to understand under what conditions such residual electromagnetic forces may change sign.

Motivated by this question Kenneth and Klich have recently shown that a mirror pair of dielectric bodies always attract [9] . Their proof is based on a detailed study of the determinant formula that enters in the leading-order semiclassical calculation of the force. Here I want to

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point out that this result follows from a general property of quantum field theory known as 'reflection positivity' [10, 11]. This extends the validity of the theorem to interacting quantum fields, arbitrary static vacuum probes, and (with assumptions to be discussed) beyond the semiclassical approximation of the probes, as well as in the presence of spectator bodies or material media. The proof applies equally well to electrostatic and magnetostatic forces, as to those arising from the quantum and thermal fluctuations of the electromagnetic field. A variant of the argument shows that torques always tend to rotate a pair of conjugate probes into a mirror-symmetric configuration. The arguments presented here have been used in the past to establish the concavity of the quark/anti-quark potential [12, 13]. The main point of the present letter is to explain that they apply to a much broader class of physical situations.

Reflection positivity is a general property of Euclidean quantum field theory which guarantees the existence of a positive Hilbert space and a self-adjoint, non-negative Hamiltonian [10]. It can be stated as the following correlation inequality:

$$\langle\, f\,\Theta(f)\,\rangle \quad \geq \quad 0\ , \tag{1}$$

where Θ denotes the reflection with respect to a three-dimensional hyperplane in R^4, f is any functional of the fields with arguments to the left of the hyperplane, and the action of Θ on f is anti-unitary [i.e. $\Theta(cf) = c^*\Theta(f)$]. Inequality (1) can be established easily in the path-integral formulation of quantum field theory. Consider for example a real scalar field defined on a hypercubic lattice Λ which admits Θ as a reflection symmetry. Let the functional-integral measure be :

$$d\mu = (\prod_x d\phi_x)\, e^{-S_\phi} \quad \text{with} \quad S_\phi = \sum_{<x,y>} \frac{1}{2}(\phi_x - \phi_y)^2 + \sum_x V(\phi_x)\ , \tag{2}$$

where V is the scalar potential and $< x, y >$ the lattice links. We also define Λ_+, Λ_0 and Λ_- to be the collections of sites and links that lie to the left, on, or to the right of the mirror hyperplane. Splitting the integration measure accordingly, $d\mu = d\mu_+ d\mu_0\, d\mu_-$, one finds

$$\langle\, f(\phi_x)\, f^*(\phi_{\Theta x})\,\rangle \;=\; \int d\mu_0\, \Big|\int d\mu_+ f(\phi_x)\Big|^2 \tag{3}$$

for any functional f that is defined entirely in Λ_+. The inequality (1), with $\Theta(f(\phi_x)) \equiv f^*(\phi_{\Theta x})$, follows now from the positivity of the

measure $d\mu_0$. This proof only applies, strictly-speaking, to lattice symmetries, but it extends to all mirror reflections in the continuum limit where rotational invariance is restored.

Reflection positivity for gauge fields can be established in a similar way [11], starting for instance from the Wilson action

$$S_{\rm Wilson} \;=\; -\frac{1}{2g^2}\sum_P \,{\rm Re\ tr}\Big(\prod_{<x,y>\in P} U_{<x,y>}\Big)\ . \qquad (4)$$

Here P labels the lattice plaquettes, the product runs over the links of P in a path-ordered way, and the link variables $U_{<x,y>} = U^\dagger_{<y,x>}$ take values in the gauge group [14]. We define the action of Θ as follows: $\Theta(f(U_{<x,y>})) \equiv f^*(U_{<\Theta x,\Theta y>})$, and note that it leaves invariant the Wilson action. Inequality (1) follows now easily from positivity of the Haar measure, $\prod dU_{<x,y>}$, by a similar argument as above. The extension to fermion fields is also straightforward, provided one defines the action of Θ appropriately [15, 11]. Thus (1) is a general property of relativistic quantum field theories, and in particular of Quantum Electrodynamics and of the Standard Model.

It is important here to emphasize that the existence of a reflection operator Θ satisfying (1) is only a consequence of unitarity, and makes no assumptions about the discrete symmetries P, C and T.[1] This is made clear by the following facts:

- A bounded-from-below potential for charged scalar fields respects reflection positivity, even though it can lead to explicit or spontaneous breaking of the charge-conjugation symmetry C.
- A θ-term respects reflection positivity, but breaks explicitly parity and time reversal. Note that $\theta\int \epsilon_{\mu\nu\rho\sigma}{\rm tr}F^{\mu\nu}F^{\rho\sigma}$ is odd under P or T, but it makes an imaginary contribution to the Euclidean action. Thus the combination of reflection and complex conjugation leaves invariant the functional-integral measure, and inequality (1) continues, indeed, to hold.[2]

In some of the situations discussed below, Θ will act in the same way as the combination CP. This is however only a coincidence, as the above examples clearly illustrate.

[1] The issue was raised by Adam Schwimmer. I am grateful to him, as well as to Henri Epstein and John Iliopoulos, for helping to clarify it.

[2] Defining the θ-term on the lattice, so as to make the argument rigorous, is tricky [16].

There is a second point that needs clarification: positivity of the theory requires, strictly-speaking, inequality (1) only for reflections in the direction of (Euclidean) time. These are indistinguishable from space reflections if we consider probes of a relativistic vacuum. In the presence of a medium, on the other hand, the measure need not be invariant under an operation Θ that reflects space. Only when it is, will we be allowed to extend the validity of our theorem.

Let us now concentrate on functionals f which, when inserted in the path integral, describe static semiclassical probes interacting with the electromagnetic vacuum. Examples are a superconducting shell, a dielectric body, or (a collection of) infinitely-heavy electric charges. The corresponding functionals read:

probe	conductor	dielectric	point charges
f	$\prod_{\mathrm{P}\in\mathcal{W}} \delta(U_P - 1)$	$\prod_{\mathrm{P}_0\in\mathcal{W}} \exp(\alpha \,\mathrm{Re}\, U_{P_0})$	$\prod_j (U_j)^{q_j}$

.

In these formulae U_P are the plaquette variables, $\mathcal{W}$ is the world-volume of the dielectric body or the conducting shell, P_0 labels time-like plaquettes (i.e. those with one link in the time direction), and the U_j are Wilson lines along the worldline of the jth particle whose electric charge is q_j. Note that $U_P \simeq \exp(iF_P)$, where F_P is the electromagnetic field strength along the plaquette. Thus the δ-functions in the first entry impose the vanishing of the electric and magnetic fields tangent, respectively normal, to the conducting shell, as appropriate. Note also that the material of the dielectric body has been assumed isotropic and non-dispersive, with relative permittivity ϵ given by $\alpha = (1-\epsilon)/2g^2$. We will comment on dispersive media later on. Note finally that one can describe geometrically-smooth probes by using linear combinations of plaquette field strengths in the definition of the above functionals.[3]

For static probes the functionals f are time-translation invariant. We may thus impose periodic boundary conditions in the time direction ($x^0 = x^0 + \beta$) so that

$$\langle f_1 \cdots f_N \rangle \;=\; e^{-\beta\, E(1,\cdots,N)} \tag{5}$$

[3]The use of lattice variables is not essential. It shows, however, that the relevant inequalities survive the (non-perturbative) regularization of the theory.

where β is the inverse temperature, and E is the free energy of the system in the presence of the probes $1, \cdots, N$. The existence of a continuum limit for the correlator (5) raises, in fact, some subtle mathematical questions. Careful subtraction procedures have been proposed for Wilson-loop operators [17] and, at the one loop-level, for perfectly conducting shells [18]. A weak assumption, verified explicitly in the above two examples, is that (5) can be made finite by a multiplicative renormalization of the functionals, $f_a^{\rm ren} = \exp(-\beta E_a)\, f_a$, where E_a is the (generally divergent) self-energy of the ath probe. This will be sufficient for our purposes here, since we will focus on the dependence of the energy on the relative position and orientation of the probes, rather than on their detailed composition and shape.

We come now to the main point of this paper. Reflection positivity implies that $(f_1, f_2) \equiv < f_1 \Theta(f_2) >$ is a non-negative inner product on the space of functionals that are defined on one side of a reflection hyperplane. This implies, in turn, the Schwarz inequality:

$$|\langle f_1 \Theta(f_2) \rangle|^2 \ \le \ \langle f_1 \Theta(f_1) \rangle \langle f_2 \Theta(f_2) \rangle \tag{6}$$

for any two probes 1 and 2 that live on the same side of the hyperplane. Consider the special case of identical probes, with the same orientation and with centers of mass aligned on the Oz axis. Let Θ act by reflecting the coordinate z (see figure 1). Combining inequality (6) with equation (5) we find :

$$2E_{\rm mir}(z_1 + z_2) \ \ge \ E_{\rm mir}(2z_1) + E_{\rm mir}(2z_2) \ , \tag{7}$$

where $E_{\rm mir}(z)$ is the interaction energy of the mirror pair of conjugate probes, and z is the separation of their centers. It follows from (7) that $E_{\rm mir}(z)$ is a concave function, i.e. its second derivative is nowhere positive. Thus $dE_{\rm mir}/dz$ is monotone non-increasing. Since, furthermore, the energy is bounded at infinite separation from below, we conclude that $dE_{\rm mir}/dz$ is everywhere non-negative, so that the force is either attractive or exactly zero. This generalizes the proof of reference [9] to arbitrary static external probes, and beyond the one-loop approximation.[4] An extra immediate corollary is that the binding energy of a pair of conjugate probes can never grow faster than linearly with distance [12].

[4]The fact that a spherical perfectly-conducting thin shell feels an outward Casimir pressure [19] does not contradict the theorem. It has indeed been recognized that separating an elastic shell into two rigid hemispheres is a mathematically-singular operation, which introduces divergent edge contributions to the energy [20, 18].

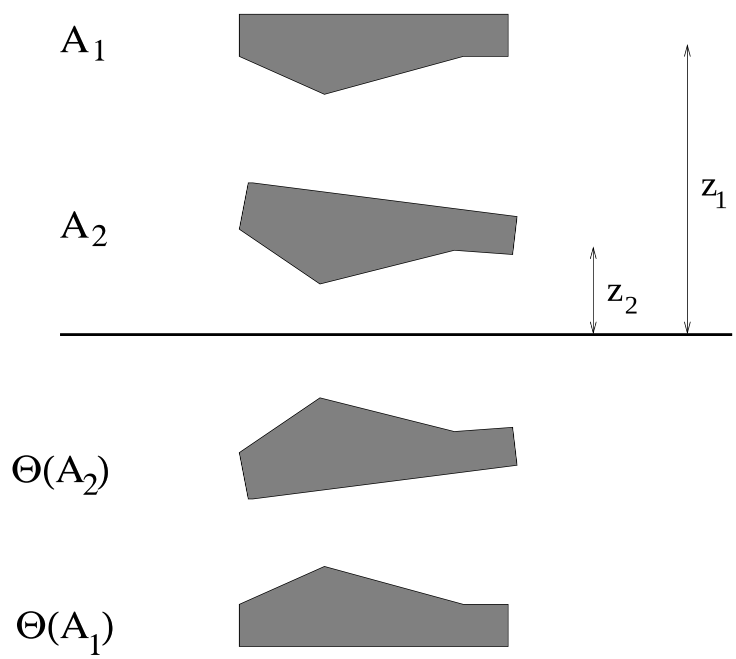

Figure 1: Two identical probes with centers of mass aligned along the Oz axis, and their mirror images in the (xy) plane. Probe 2 is in general rotated with respect to probe 1. Inequalities (7) and (10) are obtained in the special limits of zero rotation, or for $z_1 = z_2$.

Notice that it is crucial for the theorem's validity that the operation Θ involves a complex conjugation. This does not affect neutral probes, like the conducting shell or the dielectric bodies, but it does flip the sign of electrostatic forces. To illustrate the point consider the classical interaction energy between a pair of electric dipoles :

$$E(\mathbf{d}_1, \mathbf{d}_2|\mathbf{r}) \;=\; \frac{\mathbf{d}_1 \cdot \mathbf{d}_2 - 3(\mathbf{d}_1 \cdot \hat{\mathbf{r}})(\mathbf{d}_2 \cdot \hat{\mathbf{r}})}{|\mathbf{r}|^3} \;. \tag{8}$$

The action of Θ on an electric dipole is $\Theta(\mathbf{d}) = -\mathbf{d} + 2\hat{\mathbf{n}}(\mathbf{d}\cdot\hat{\mathbf{n}})$, where $\hat{\mathbf{n}}$ is a unit normal to the reflection plane, and the overall minus sign comes from charge conjugation. Setting $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ gives the interaction energy between a mirror pair of conjugate dipoles

$$E(\mathbf{d}, \Theta(\mathbf{d})|\mathbf{r}) \;=\; -\frac{\mathbf{d}\cdot\mathbf{d} + (\mathbf{d}\cdot\hat{\mathbf{r}})^2}{|\mathbf{r}|^3} \;. \tag{9}$$

The force between the pair is thus attractive, consistently with reflection positivity. Note that for a magnetic dipole created by a persistent current $\Theta(\mathbf{m}) = \mathbf{m} - 2\hat{\mathbf{n}}(\mathbf{m}\cdot\hat{\mathbf{n}})$. The theorem nevertheless continues to hold, because the interaction energy of two such magnetic dipoles is minus the expression in (8), see for example [21] .

As a further application of inequality (6) we can consider the dependence of the energy on the spatial orientation of the probes. Let 1 and 2 be obtained from some reference probe by rotations $\mathcal{R}_1$ and $\mathcal{R}_2$ around its center of mass (see figure 1). Inequality (6) then reads in a self-explanatory notation

$$2E(\mathcal{R}_1, \Theta(\mathcal{R}_2)) \;\geq\; E(\mathcal{R}_1, \Theta(\mathcal{R}_1)) + E(\mathcal{R}_2, \Theta(\mathcal{R}_2)) \;. \tag{10}$$

It follows that $E(\mathcal{R}_1, \Theta(\mathcal{R}_2))$ is greater than the smaller of the two terms on the right-hand side, so that the minimum of the energy is obtained when the relative orientation of conjugate probes corresponds precisely to that of a mirror pair.[5] Since in this configuration the force is, as we saw, attractive, such systems will always tend to collapse unless stabilized by external torques and forces.

Let us now turn attention to situations in which a material medium, or a spectator body, break Lorentz invariance even before the insertion of external probes. Spatial and (Euclidean-) time reflections are no

[5]The absolute preferred orientation depends on the details of the probes, and cannot be determined by such general arguments.

more equivalent, so that the theorem need not always hold. Its (partial) validity requires that the functional measure be invariant under (at least a subset of) *spatial* reflections and complex conjugation. We have already used this fact above, in our discussion of the thermal vacuum. The following examples further illustrate the point:

- Two identical pistons inside an infinite conducting cylinder will attract, in agreement with the conclusion of references [22]. This follows from Θ-invariance for all reflections that are geometric symmetries of the cylinder.
- The proof continues to apply in the presence of a classical homogeneous and isotropic dispersive medium, as discussed already by Kenneth and Klich [9]. The Euclidean Maxwell action in this case can be written as:

$$S_{\text{Max}} = \int d^3\mathbf{x} \int_0^\infty \frac{d\omega}{2\pi} \left(\epsilon(i\omega)|\mathbf{E}(\omega,\mathbf{x})|^2 + |\mathbf{B}(\omega,\mathbf{x})|^2 \right) \ , \qquad (11)$$

  where $\epsilon(\omega)$ is the dielectric permittivity, which must be analytic in the upper-half complex plane, and real and positive on the imaginary ω axis [23]. Although this action is non-local in Euclidean time, it does satisify reflection positivity for space reflections, so that a mirror pair of conjugate probes still attracts.[6]
- A case where the theorem doesn't hold, consistently with claims in the literature [24], is that of mirror probes in a Fermi sea. Indeed a non-zero chemical-potential term, μj^0 where j^0 is the relevant number density, is invariant under charge conjugation and time reflection, but not under charge conjugation and *space* reflection. This seems to contradict our first example, since a conducting material could be after all modeled by a free-electron gas. The point is, however, the following: the theorem holds to the extent that the conductor can be described by a classical permittivity for the photon field, but doesn't apply if the electron-gas fluctuations become important.[7]

This last point can be made more clear by considering the extension of the theorem to quantum probes. A simple example is a charged

[6]Note that the dielectric or superconducting probes discussed so far can be thought of as special limits of a spatially-varying permittivity $\epsilon(\omega,\mathbf{x})$.

[7] Other examples that violate the assumptions of our theorem are periodic boundary conditions for fermions [15], or the situations discussed in reference [25].

non-relativistic quantum particle trapped inside a potential well. The corresponding functional of the electromagnetic field (that replaces the Wilson line of a classical charge) reads :

$$f(A^\mu) = \int [D\mathbf{x}(t)]\, e^{-I}\, e^{iq \int (A^0 dt - \mathbf{A}\cdot d\mathbf{x})}\ . \tag{12}$$

Here $[D\mathbf{x}(t)]$ is the integration measure over particle trajectories,

$$I \ = \int_0^\beta dt\ \left[\frac{m}{2}(\frac{d\mathbf{x}}{dt})^2 + V(\mathbf{x})\right] \tag{13}$$

is the Euclidean non-relativistic particle action, and $V(\mathbf{x})$ the localizing potential. Plugging the functionals (12) in equation (5) gives the free energy of the combined system of particles and fields at inverse temperature β. If we could use inequality (6), we would then conclude as before that the force between a pair of conjugate probes is attractive. The problem with this argument is that the quantized particle does not, strictly-speaking, live in one side of the reflecting plane. Forcing it to do so (e.g. by adding to V an artificial hard-wall term) will, however, introduce only a tiny error if the particle-mirror separation is much bigger than the localization length. For a harmonic-well potential the error will, for instance, decay exponentially fast. For sufficiently-large probe separations the conclusions of the theorem should thus still apply.

We can easily extend the above argument to collections of particles, such as the protons, neutrons and electrons of a heavy atom. Note, however, that because Θ acts as charge conjugation, the mirror partner must be made of antimatter! Thus this last extension of the theorem to truly microscopic quantum probes is of little relevance for applications to realistic electromechanical systems.

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